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Effective volume of specimens in diametral compression

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A calculation was performed to find the effective tensile volume for diametral compression tests of brittle materials. Both disc and spherical geometries were investigated. The effective volume equation for disc specimens separates into materials-dependent and geometry-dependent functions, which are separately solved. The effective volume equation for spherical specimens is mathematically more difficult and less enlightening.

1. Introduction

Diametral compression is used to measure tensile failure in disc-shaped brittle specimens [1, 2]. One typical application is the quantification of concrete degradation using slices of a drilled core. Diametral compression does not require extensive machining of specialized shapes and it is self-aligning. One disadvantage is that the stress state, while known for some time [3], is complex. This makes it difficult to compare data from a diametral compression test to data from a uniaxial tension test or a flexure test. More recently, spheres have been used in diametral compression [4]. The stress state in a compressed sphere is similarly known and complex.

The purpose of this work is to report an effective uniaxial volume under load for diametral compression tests. Both disc and spherical geometries are considered.

The failure stress of a brittle material is related to the stress applied, and the size, position and orientation of the most severe flaw. This may be treated using weakest-link statistics to provide the probability of failure as a function of stress. The resulting equation follows [5].

$$P = 1 - \exp\left(\frac{-1}{V_o}\int_V k_o(m, \sigma_1, \sigma_2, \sigma_3) \times \left(\frac{\sigma_1(xyz)}{\sigma_o}\right)^m \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right) \quad (1)$$

where P = cumulative probability of failure σ_o = normalizing strength, one of the Weibull parameters; m = Weibull modulus, another Weibull parameter; V = specimen volume; V_o = unit volume, or volume under stress during test; $k_o(m, \sigma_1, \sigma_2, \sigma_3)$ = a stress state function; and $\sigma_1, \sigma_2, \sigma_3$ = principle stresses.

For a specimen under uniform uniaxial tensile stress, the preferred testing technique, the equation simplifies considerably.

$$P = 1 - \exp\left(-\left(\frac{\sigma_1}{\sigma_o}\right)^m\right) \quad (2)$$

Under uniaxial stress, k_o is one. A uniform stress field does not require integration over the volume. During the test, $V = V_o$. Testing techniques with non-uniform stress are often used instead of uniform tensile testing because they are experimentally simpler. Data from these techniques may be used in the same way as data from uniform tensile testing with the difference that V_{eff} (an effective volume under stress) is substituted for V_o. The stress applied is generally taken as the maximum stress in a non-uniform sampling technique. It is generally best to sample the largest volume possible for good statistics, meaning that more uniform stress states are more desirable.

Ceramics generally fail in pure tension; therefore compressive stresses do not contribute to the probability of failure. Calculations may be simplified by assuming all compressive stresses to have zero magnitude.

The material specific parameters necessary to make design predictions are σ_o, m and V_o or V_{eff}. These values, coupled with the internal stresses of a known design allow calculation of the probability of failure. If diametral compression is to be used to make design decisions, the effective volume must be known.

In a thin disc specimen subjected to diametrically opposed in-plane line loads, the stress is assumed uniform across the thickness. The stress is therefore two-dimensional. The vertical stress is compressive (negative) everywhere; therefore it is ignored. The horizontal stress perpendicular to the line of applied force (σ_h) is tensile throughout most of the volume (see Fig. 1). It exhibits a stress singularity near the point forces, but this singularity is compressive, and may be ignored. The functional form of σ_h is

$$\sigma_h = \frac{2P}{\pi D t}\left\{1 - \frac{D}{a_1}\sin^2\theta_1\cos^2\theta_1 - \frac{D}{a_2}\sin^2\theta_2\cos^2\theta_2\right\} \tag{3}$$

where P/t = line force applied; t = disc thickness; D = disc diameter; a = vertical distance from point of application of line force; and θ = angle from vertical. Subscripts 1, 2 refer to the points of application of forces 1, 2.

The coordinate system is shown diagrammatically in Fig. 2. The preceding formula is taken from [3], Equation 66, modified by the discussion in Section 41.

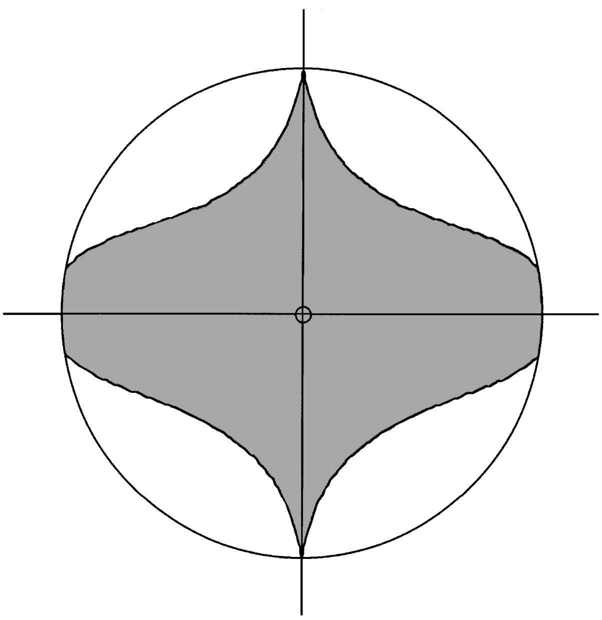

Figure 1 Distribution of uniaxial tensile stress in a diametrically compressed disc specimen.

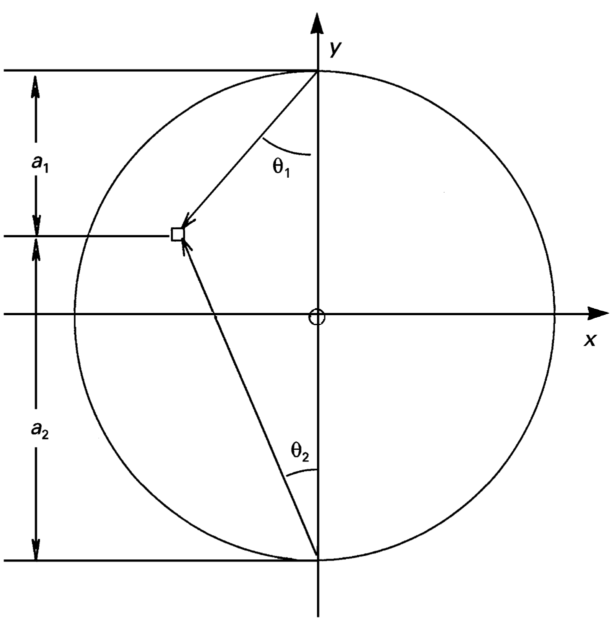

Figure 2 Coordinate system for a disc in diametral compression.

The equation is the juxtaposition of three stress fields, one generated by each point force and a uniform tensile stress to ensure that the edge of the disc is stress free. This stress maximum is $2P/\pi Dt$ and occurs at the loaded diameter, where $\theta = 0$. Most disc specimens in diametral compression fail near this diameter.

Exceptions occur when the ceramic fails in shear (Hertzian contact failure) near the point of load application. The fragments due to failure in shear differ greatly from the fragments due to failure in tension. Specimens that fail in shear may therefore be excluded from the statistical sample.

The stress state in spherical specimens under diametrically opposed point forces is similarly known. The system is cylindrically symmetrical. The z stress component is compressive (negative) everywhere; therefore it is ignored. The radial stress perpendicular to the line of applied force (σ_r) is tensile throughout much of the volume (see Fig. 3). The extent of the tensile stress is not a function of Poisson's ratio. The functional form of σ_r is

$$\begin{aligned}\sigma_r = \frac{P}{2\pi R^2}\Bigg\{&0.75 + (1-2\nu)\frac{R^2}{r^2}\\ &\times\left[2 - \frac{z}{(r^2+z^2)^{0.5}} - \frac{(2R-z)}{[r^2+(2R-z)^2]^{0.5}}\right]\\ &- \frac{3r^2R^2z}{(r^2+z^2)^{2.5}} - \frac{3r^2R^2(2R-z)}{(r^2+(2R-z)^2)^{2.5}}\Bigg\}\end{aligned} \tag{4}$$

where R = the radius of the sphere.

The hoop stress, σ_θ, is tensile throughout the volume of the sphere. The functional form of σ_θ is

$$\begin{aligned}\sigma_\theta = \frac{P}{2\pi R^2}\Bigg\{&0.75 + \frac{R^2(1-2\nu)}{r^2}\\ &\times\left[2 - \frac{z}{(r^2+z^2)^{0.5}} - \frac{(2R-z)}{[r^2+(2R-z)^2]^{0.5}}\right]\\ &+ \frac{R^2(1-2\nu)z}{(r^2+z^2)^{1.5}} + \frac{R^2(1-2\nu)(2R-z)}{(r^2+(2R-z)^2)^{1.5}}\Bigg\}\end{aligned} \tag{5}$$

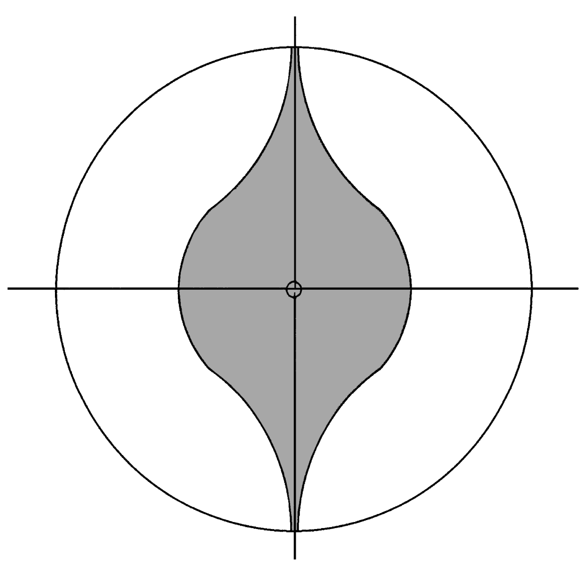

Figure 3 Extent of radial tensile stress in a compressed spherical specimen.

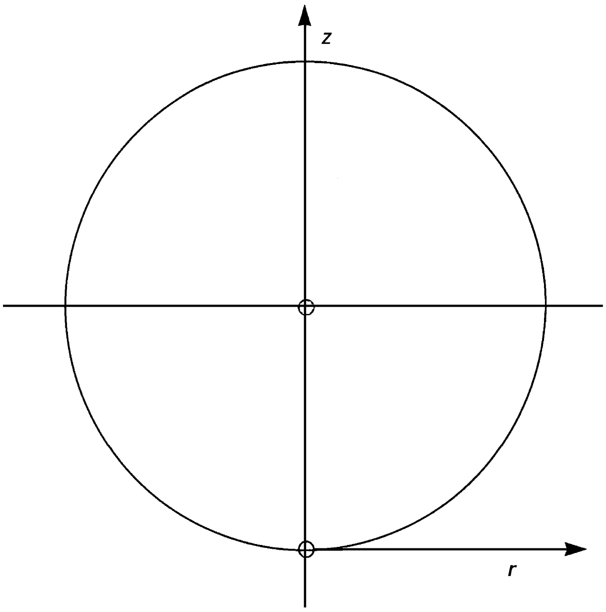

Figure 4 Coordinate system for a compressed sphere.

The preceding formulae are taken from [6], Equation 211. The origin is the point of application of one force. The other force is applied at $z = 2R$, see Fig. 4.

2. Approach

Before integrating over the disc volume using Equation 3, we shift the origin to the centre of the circle and normalize it so we integrate over the unit circle. First, we make the following substitutions into Equation 3:

$$\cos\theta_1 = \frac{1-y}{[(1-y)^2 + x^2]^{0.5}}$$

$$\sin\theta_1 = \frac{x}{[(1-y)^2 + x^2]^{0.5}}$$

$$\cos\theta_2 = \frac{1+y}{[(1+y)^2 + x^2]^{0.5}}$$

$$\sin\theta_2 = \frac{x}{[(1-y)^2 + x^2]^{0.5}}$$

$$\frac{1}{a_1} = \frac{2}{(1-y)D}$$

$$\frac{1}{a_2} = \frac{2}{(1+y)D}$$

x and y are normalized with respect to the radius of the disc, and define distances from the centre.

After substitution, Equation 2 has the form

$$\sigma_h = \frac{2P}{\pi D t}\left\{1 - \frac{2x^2(1-y)}{(x^2 + (1-y)^2)^2} - \frac{2x^2(1+y)}{(x^2 + (1+y)^2)^2}\right\} \tag{6}$$

The part of the failure equation inside the exponential parentheses is the risk of rupture: B. We are looking for a relationship for V_{eff} so that B for the disc equals B of a uniaxial tensile sample of volume V_{eff} under the same stress σ. That is,

$$\frac{-1}{V_0}\int_V \left(\frac{\sigma_1(xyz)}{\sigma_0}\right)^m \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{V_{eff}}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m$$

When we substitute Equation 6 into Equation 1, we get

$$B = \frac{-1}{V_0}\int_V \left[\frac{2P}{\pi D t \sigma_0}\left\{1 - \frac{2x^2(1-y)}{(x^2 + (1-y)^2)^2} - \frac{2x^2(1+y)}{(x^2 + (1+y)^2)^2}\right\}\right]^m \mathrm{d}x\,\mathrm{d}y$$

If we then define $\sigma = 2P/\pi D t$, and rearrange, we get

$$B = \frac{-1}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m \int_V \left\{1 - \frac{2x^2(1-y)}{(x^2 + (1-y)^2)^2} - \frac{2x^2(1+y)}{(x^2 + (1+y)^2)^2}\right\}^m \mathrm{d}x\,\mathrm{d}y$$

or

$$V_{eff} = D^2 t \int_0^1 \int_0^{(1-x^2)^{0.5}} \left\{1 - \frac{2x^2(1-y)}{(x^2 + (1-y)^2)^2} - \frac{2x^2(1+y)}{(x^2 + (1+y)^2)^2}\right\}^m \mathrm{d}x\,\mathrm{d}y \tag{7}$$

The integral is a function of m only, and may be called $N(m)$. The positive (tensile) portion of $N(m)$ has been numerically integrated for m ranging from 1 to 30 in increments of 0.1. These values appear in the Appendix. The function is a smoothly decreasing function and may be approximated by the function

$$N(m) = 0.21209\, m^{-0.48338}$$

This approximation is accurate to better than 2% over the range calculated.

Before integrating over the volume of a sphere using Equations 4 and 5, we shift the origin to the centre of the circle and normalize it. First, we make the following substitutions into Equations 4 and 5;

$$z = R(1-y)$$

$$(2R - z) = R(1+y)$$

$$r = Rx$$

After substitution, Equation 4 has the form

$$\sigma_r = \frac{P}{2\pi R^2}\left\{0.75 + \frac{(1-2\nu)}{x^2}\left[2 - \frac{(1-y)}{[x^2 + (1-y)^2]^{0.5}} - \frac{(1+y)}{[x^2 + (1+y)^2]^{0.5}}\right] - \frac{3x^2(1-y)}{(x^2 + (1-y)^2)^{2.5}} - \frac{3x^2(1+y)}{(x^2 + (1+y)^2)^{2.5}}\right\} \tag{8}$$

Equation 5 has the form

$$\sigma_\theta = \frac{P}{2\pi R^2}\left\{0.75 + \frac{(1-2\nu)}{x^2}\left[2 - \frac{(1-y)}{[x^2 + (1-y)^2]^{0.5}} - \frac{(1+y)}{[x^2 + (1+y)^2]^{0.5}}\right] + \frac{(1-2\nu)(1-y)}{(x^2 + (1-y)^2)^{1.5}} + \frac{(1-2\nu)(1+y)}{(x^2 + (1+y)^2)^{1.5}}\right\} \tag{9}$$

When integrating these formulae over the unit circle, note that the relation

$$\frac{1}{x^2}\left[2-\frac{(1-y)}{[x^2+(1-y)^2]^{0.5}}-\frac{(1+y)}{[x^2+(1+y)^2]^{0.5}}\right] \tag{10}$$

in Equations 8 and 9 is undefined for $x=0$. In the limit $x\to 0$, it does however converge to

$$\left[\frac{1}{2(1-y)^2}+\frac{1}{2(1+y)^2}\right] \tag{11}$$

The numerical integration of the effective volume for a spherical sample is the sum of three integrations. One is near $x=0$ using Equation 11 calculating for a biaxial stress state. The second is over the volume in biaxial tension, away from $x=0$. The third is over the volume under uniaxial tension. The stress state is biaxial along the stressed diameter.

Two approaches present themselves for calculating the effective volume under biaxial stress. The first is to use the principle of independent action (PIA) [7], assuming the probability of failure is the sum of probabilities of failure for each principle stress. Alternatively, one could apply the normal stress averaging (NSA) theory [5], integrating about the unit circle for each point to find the probability of failure. It is not useful to apply Batdorf theory [8]. Because of the extremely high Hertzian contact stresses, if the material is shear-sensitive, it will fail in shear, and not be a valid representation of a uniaxial stress state. For non-shear sensitive materials, Batdorf theory simplifies to NSA theory.

Effective volume for a spherical compression specimen is a function of three variables, whereas the disc was a function of one variable only. The internal stresses in a sphere are a function of Poisson's ratio. Granted, Poisson's ratio does not vary much for most engineering materials, but a separate table would be needed for each value of Poisson's ratio if the data are presented in tabular form. A more serious problem is the tensile singularity near the points of applied force. When integrated over the volume of the sphere, this function leads to an effective volume of zero. Spheres of real materials placed in diametral compression deform, and the force is not a point force. The function can therefore be integrated to some y less than one, avoiding the singularity. This is a mathematically simple and physically reasonable approximation. The effective volume of a sphere under diametral compression is therefore a function of the Weibull exponent, Poisson's ratio and the distance the sphere is compressed. Unless Equation 5 can be solved in closed form, this makes the solution intractable. Even if it is, measuring the deformation distance adds a level of experimental complexity.

In addition to adding to the computational difficulty, the high contact stress decreases the effective volume. Consider a typical engineering material under 1% deformation with a Weibull modulus of 10. The average stress in the 0.01 of the volume near the contact point is 5000 times that in the rest of the sample, leading to an effective volume less than 10^{-5}.

3. Summary

The stress field in a compressed disc specimen is integrated to find the effective tensile volume. The resulting effective volumes are monotonically decreasing with increasing *m*. The results are tabulated in the Appendix. A method is outlined for performing a similar calculation on spherical specimens. It is concluded, however, that such calculations are best done on an as-needed basis.

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Appendix: See page 2533

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Appendix: normalized effective volume for a range of Weibull exponents

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.21209	0.203495	0.195819	0.188912	0.182661	0.17697	0.171763	0.166976	0.162558	0.15846
2	0.154656	0.151105	0.147783	0.144666	0.141735	0.138973	0.136363	0.133893	0.131551	0.12932
3	0.127208	0.125191	0.123265	0.121425	0.119663	0.117976	0.116358	0.114803	0.113309	0.11187
4	0.110487	0.109152	0.107864	0.10662	0.105418	0.104256	0.103131	0.102041	0.100986	0.09996
5	0.098968	0.098003	0.097066	0.096155	0.09527	0.094408	0.093569	0.092752	0.091956	0.09118
6	0.090423	0.089685	0.088964	0.088261	0.087573	0.086902	0.086246	0.085604	0.084977	0.08436
7	0.083762	0.083174	0.082598	0.082033	0.08148	0.080939	0.080407	0.079886	0.079375	0.07887
8	0.078382	0.077899	0.077425	0.07696	0.076502	0.076053	0.075612	0.075178	0.074752	0.07433
9	0.07392	0.073514	0.073115	0.072723	0.072337	0.071956	0.071582	0.071214	0.070851	0.07049
10	0.070141	0.069795	0.069453	0.069116	0.068784	0.068457	0.068134	0.067816	0.067503	0.06719
11	0.066888	0.066587	0.06629	0.065997	0.065708	0.065422	0.065141	0.064862	0.064588	0.06431
12	0.064049	0.063784	0.063523	0.063265	0.06301	0.062758	0.062509	0.062263	0.06202	0.06178
13	0.061543	0.061308	0.061076	0.060846	0.060619	0.060395	0.060173	0.059954	0.059736	0.05952
14	0.059309	0.059099	0.058891	0.058685	0.058481	0.05828	0.05808	0.057883	0.057687	0.05749
15	0.057302	0.057113	0.056925	0.056739	0.056555	0.056372	0.056192	0.056013	0.055836	0.05566
16	0.055486	0.055314	0.055143	0.054974	0.054807	0.054641	0.054476	0.054313	0.054151	0.05399
17	0.053832	0.053675	0.053519	0.053365	0.053211	0.053059	0.052908	0.052759	0.052611	0.05246
18	0.052318	0.052174	0.05203	0.051888	0.051747	0.051608	0.051469	0.051331	0.051195	0.05105
19	0.050925	0.050792	0.050659	0.050528	0.050398	0.050269	0.05014	0.050014	0.049887	0.04976
20	0.049638	0.049514	0.049392	0.04927	0.049149	0.049029	0.04891	0.048792	0.048675	0.04855
21	0.048443	0.048328	0.048214	0.048101	0.047989	0.047877	0.047766	0.047656	0.047547	0.04743
22	0.047331	0.047223	0.047117	0.047012	0.046907	0.046802	0.046699	0.046596	0.046494	0.04639
23	0.046291	0.046191	0.046092	0.045993	0.045895	0.045797	0.0457	0.045603	0.045508	0.04541
24	0.045317	0.045223	0.04513	0.045037	0.044945	0.044853	0.044762	0.044671	0.044581	0.04449
25	0.044403	0.044314	0.044226	0.044139	0.044052	0.043966	0.04388	0.043795	0.04371	0.04362
26	0.043541	0.043458	0.043375	0.043293	0.043211	0.043129	0.043048	0.042967	0.042887	0.04280
27	0.042728	0.042649	0.042571	0.042493	0.042416	0.042338	0.042262	0.042185	0.04211	0.04203
28	0.041959	0.041884	0.04181	0.041736	0.041663	0.04159	0.041517	0.041445	0.041373	0.04130
29	0.04123	0.041159	0.041089	0.041018	0.040949	0.040879	0.04081	0.040741	0.040673	0.04060
30	0.040537									